Fast and robust procedures in high-dimensional variable selection

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Joint works with M. Bogdan (University of Wrocław) and K. Furmańczyk (Warsaw University of Life Sciences)

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- Probit model

$$P(Y_i = 1|X_i) = \Phi(\beta' X_i)$$

Variable selection

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- Under mild assumptions it can be recognized up to positive multiplicative constant

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- ... and computationally fast

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$$\arg\min_{\theta} \quad \frac{1}{n}\sum_{i=1}^{n}\left|Y_i - \theta' X_i\right| + \lambda\,|\theta|_1$$
- Time-consuming when $n, p \sim 1000$

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- Treat Y_i's as numeric and apply Lasso for linear regression

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- Bühlmann, vd Geer (2011). *Statistics for High-Dimensional Data: Methods, Theory and Applications*

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- $H = \mathbb{E}X_1 X_1'$ - positive definite

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- Using thresholding or weighted (adaptive) procedures

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- U-statistics theory is used

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- $n \geq K_1 p_0^2 \log(p/a)$
- $\lambda \geq K_2 \sqrt{\frac{\log(p/a)}{n}}$
- With probability at least $1 - K_3 a$ we have

$$|\hat{\theta} - \theta^0|_q \leq K_4 p_0^{1/q} \lambda$$

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- $p \in \{100, 400, 900, 1600\}$

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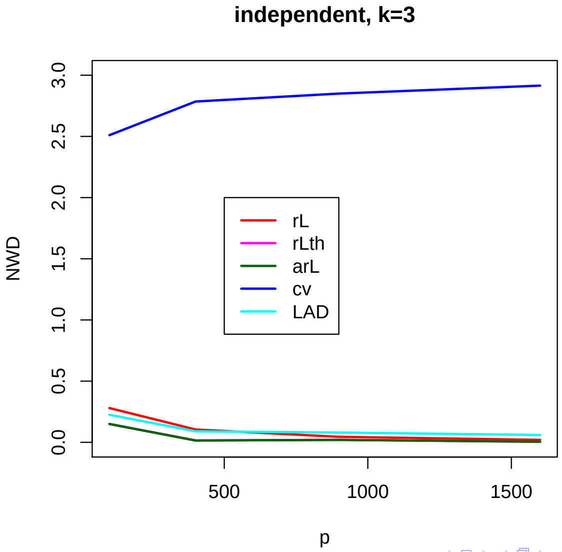
- Lasso with cross-validation (cv)

Estimators

- NWD - average number of wrong decisions


independent, k=3
NWD
0.0
0.5
1.0
1.5
2.0
2.5
3.0
500
1000
1500
p
rL
rLth
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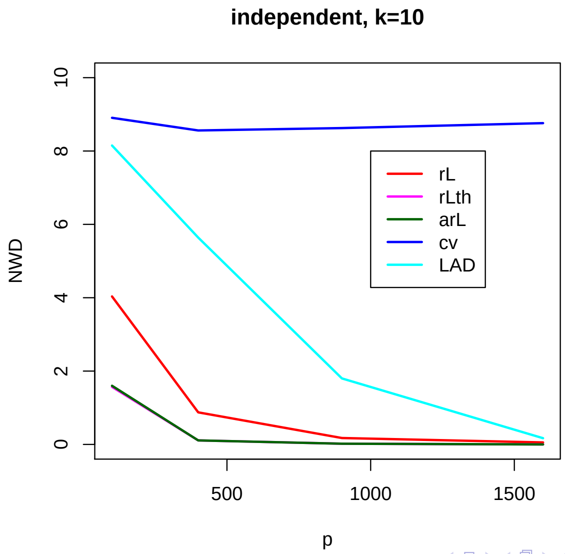
independent, k=10
NWD
0
2
4
6
8
10
500
1000
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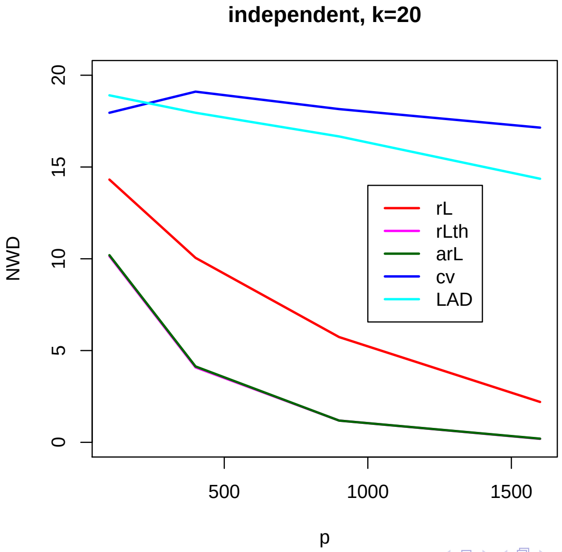
independent, k=20
NWD
p
0
5
10
15
20
500
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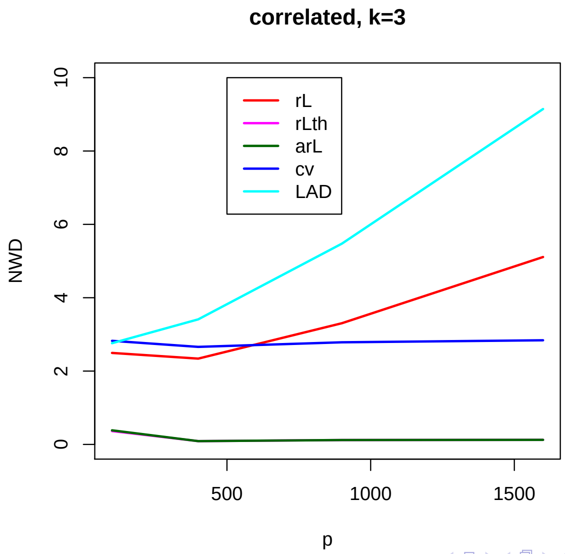
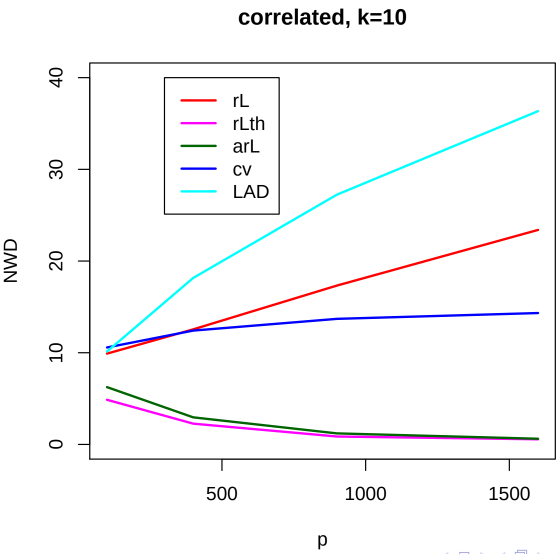
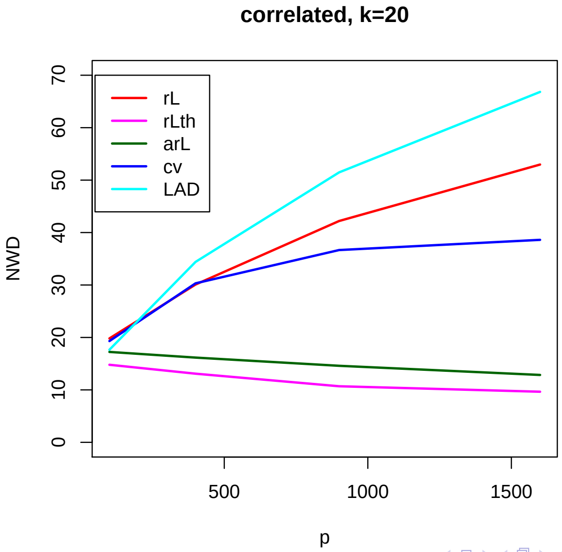

correlated, k=3
rL
rLth
arL
cv
LAD
NWD
p
0
2
4
6
8
10
500
1000
1500


correlated, k=10
rL
rLth
arL
cv
LAD
NWD
0
10
20
30
40
500
1000
1500
p


correlated, k=20
rL
rLth
arL
cv
LAD
NWD
0
10
20
30
40
50
60
70
500
1000
1500
p

Exponential model

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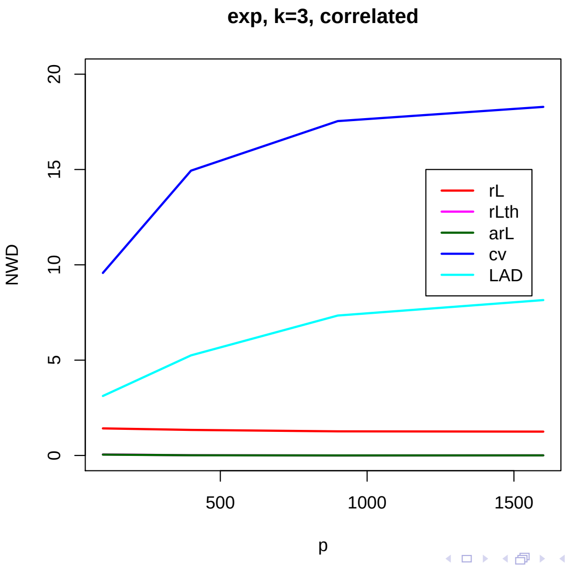
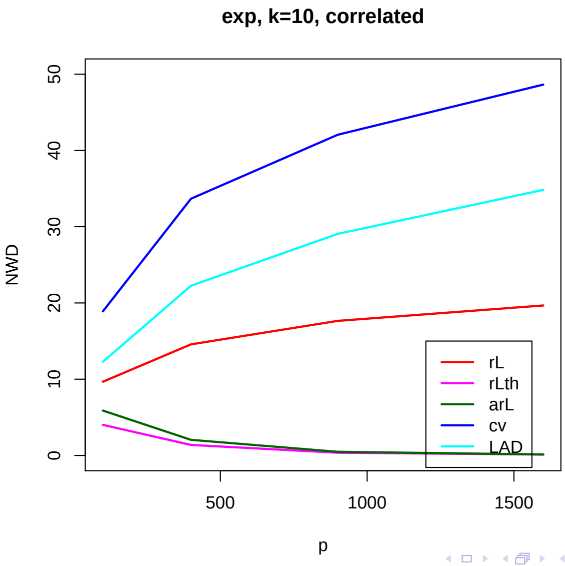
- $n \in \{100, 200, 300, 400\}$
- $p \in \{100, 400, 900, 1600\}$


exp, k=3, correlated
NWD
0
5
10
15
20
p
500
1000
1500
rL
rLth
arL
cv
LAD


exp, k=10, correlated
NWD
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10
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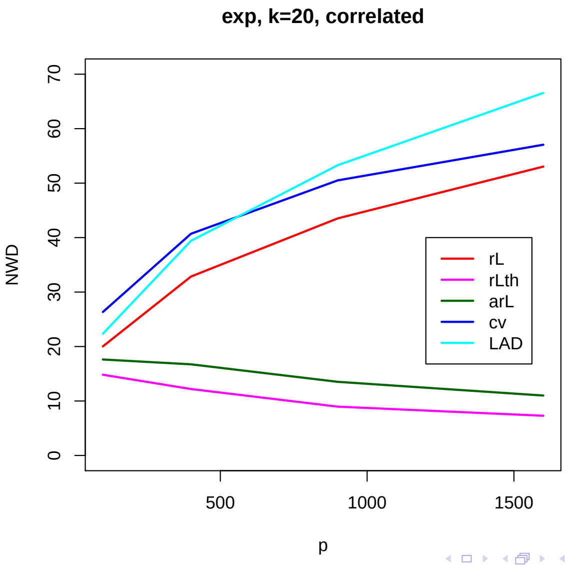
1500
p
rL
rLth
arL
cv
LAD


exp, k=20, correlated
NWD
0
10
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30
40
50
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500
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Binary regression

- $$P(Y_i = 1|X_i) = \frac{\exp(\beta' X_i)}{\exp(\beta' X_i) + 1}$$

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Estimators

- Lasso with $Y_i \in \{0, 1\}$

$$\min_{\theta} \quad \frac{1}{n} \sum_{i=1}^{n} \left(Y_i - \theta' X_i \right)^2 + \lambda \left| \theta \right|_1$$

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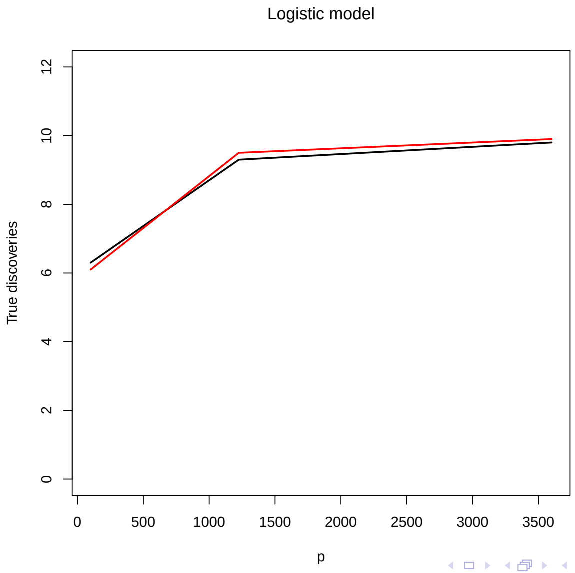
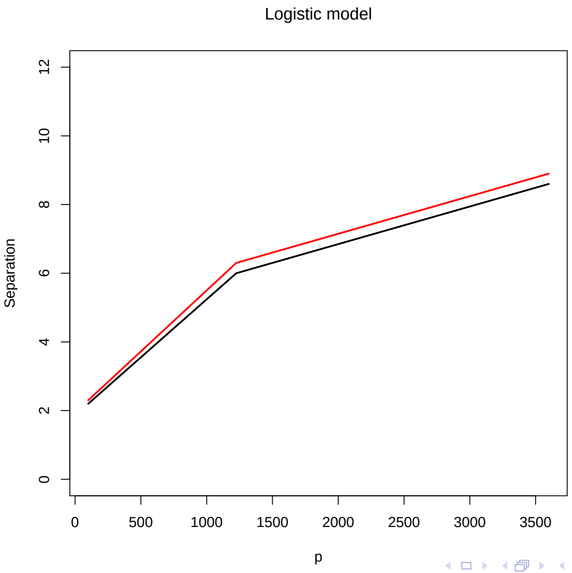
  λ chosen by CV

- Lasso for logistic regression with λ chosen by CV


Logistic model
True discoveries
0
2
4
6
8
10
12
0
500
1000
1500
2000
2500
3000
3500
p


Logistic model
Separation
0
2
4
6
8
10
12
0
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p

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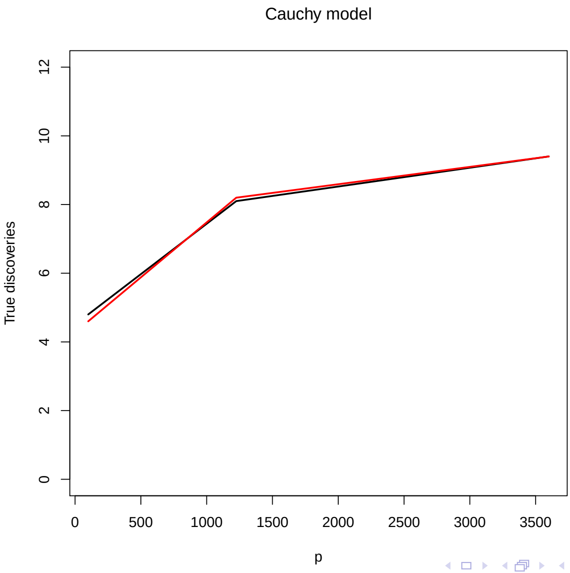
Cauchy model
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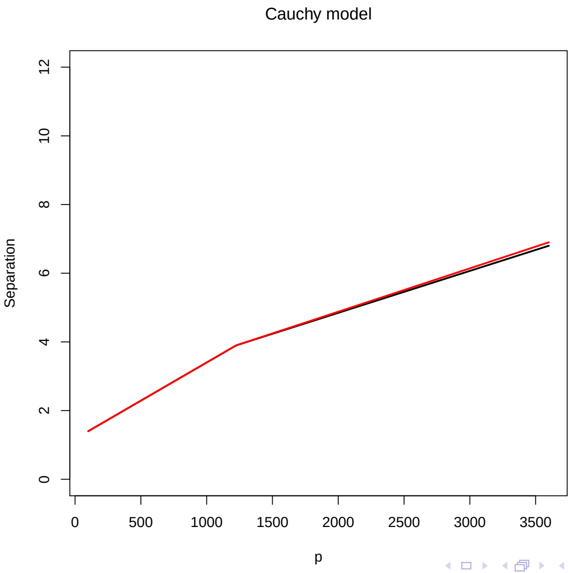
Cauchy model
Separation
p
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3500

References

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